

Uncertainty Modeling for Robust Optimization

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Motivation

Challenges in real-life applications include:

- handling of uncertain parameters
 - incomplete information
 - high-dimensional uncertainties
- robust optimization
 - bilevel formulation
 - extra effort to account for robustness
 - black box objective functions



- 1 Robust optimization
- 2 Potential clouds
- 3 Worst-case search
- 4 Applications
- 5 Summary



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Problem formulation

$$\min_{\theta \in \mathbf{T}} \max_{\varepsilon \in \mathcal{C}} g(\theta, \varepsilon) \quad (1)$$

- bilevel problem
- $\varepsilon \in \mathcal{C} \subseteq \mathbb{R}^n$ represents the uncertainties
- search space \mathbf{T} , typically a hyperinterval, possibly mixed integer variables
- black box objective function $g : \mathbf{T} \times \mathcal{C} \rightarrow \mathbb{R}$



Reformulation

Reformulate as two nested 1-level problems:

- outer level problem

$$\min_{\theta \in \mathbf{T}} \widehat{g}(\theta), \quad (2)$$

and the

- inner level problem

$$\widehat{g}(\theta) := \max_{\varepsilon \in \mathcal{C}} g(\theta, \varepsilon), \quad (3)$$

→ **worst-case search**

→ **uncertainty modeling**



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Potential clouds

- n -dimensional random vector ε
- potential function $V : \mathbb{R}^n \rightarrow \mathbb{R}$

Construct

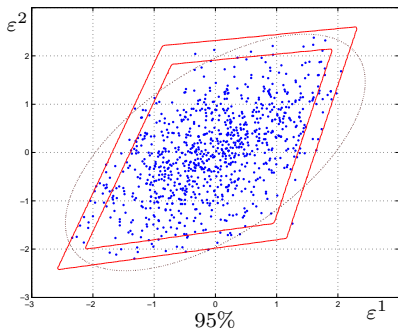
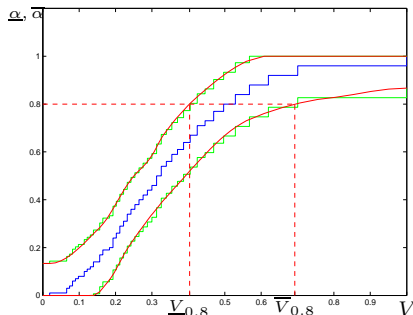
- lower α -cut $\underline{C}_\alpha := \{x \in \mathbb{R}^n \mid V(x) \leq \underline{V}_\alpha\}$
contains at most a fraction of α of all possible scenarios
 - upper α -cut $\overline{C}_\alpha := \{x \in \mathbb{R}^n \mid V(x) \leq \overline{V}_\alpha\}$
contains at least a fraction of α of all possible scenarios
- \Rightarrow nested regions defining a **potential cloud**

How?

- regard $V(\varepsilon)$ as a 1-dimensional random variable
- find an enclosure of the CDF of $V(\varepsilon)$ (p-box)



Example



- level sets of V chosen polyhedral shaped
- α -cuts reasonably approximate the confidence regions linearly



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Challenges

The worst-case search is a 1-level problem for fixed $\theta \in \mathbf{T}$ and $f(\varepsilon) := g(\theta, \varepsilon)$:

$$\max_{\varepsilon \in \mathcal{C}} f(\varepsilon). \quad (4)$$

- replaces nominal objective function of the non-robust formulation
- produces overhead of treating uncertainties
- g can be computationally expensive,
 ε can be high-dimensional
- worst-case search can be prohibitively expensive
- ⇒ speed-up required

Polyhedral uncertainty

- $\mathcal{C} := \{\varepsilon \mid A(\varepsilon - m) \leq b\}$, a polyhedral α -cut

\Rightarrow worst-case search turns into

$$\begin{aligned} \max_{\varepsilon} \quad & f(\varepsilon) \\ \text{s.t.} \quad & A(\varepsilon - m) \leq b. \end{aligned} \tag{5}$$

- classical approach: linearize f and solve LP
- simulation based approach for bound constraints $\varepsilon \in b_0$ and linear f : **Cauchy deviates method**

$$\begin{aligned} [\min_{\varepsilon} f(\varepsilon), \max_{\varepsilon} f(\varepsilon)] \\ \text{s.t.} \quad \varepsilon \in [b_c - b_r, b_c + b_r]. \end{aligned} \tag{6}$$

Modified Cauchy deviates method

- 1 evaluation at center
 - compute $f(m)$
- 2 sample \mathcal{C} uniformly
 - rejection step
- 3 transformation to Cauchy distribution via inverse Cauchy CDF with parameter b_r^i
 - Cauchy distributed sample point x_i possibly outside $\{\mathcal{C} - m\}$
- 4 normalization step
 - $K_i = \max\left(\frac{Ax_i}{b}\right) \Rightarrow \frac{x_i}{K_i} \in \{\mathcal{C} - m\}$
- 5 simulated deviation
 - $\delta_i = K_i(f_i - f(m))$, with $f_i := f\left(\frac{x_i}{K_i} + m\right)$
- 6 thus generate N sample points $\delta_1, \dots, \delta_N$



Approximation

Let $f(m + \delta) = f(m) + c^T \delta$, c unknown

$$\Rightarrow \begin{cases} 1. & \max_{\delta \in [-b_r, b_r]} f(m + \delta) = f(m) + b_r^T |c|, \\ 2. & \delta_i = c^T x_i \\ & \rightarrow \delta_i \text{ is Cauchy dist. with parameter } b_r^T |c| \end{cases}$$

\Rightarrow use the generated sample to estimate the solution statistically



Estimation

- estimate the deviation Δ from $f(m)$ via max-likelihood using $\delta_1, \dots, \delta_N$
- approximate solution

$$\max_{\varepsilon \in \mathcal{C}} f(\varepsilon) \approx f(m) + \Delta \quad (7)$$

- tractable estimation error even for very small N , useful if linearization cannot be afforded

N	800	200	100	50	20	10	5
error	10%	20%	30%	40%	70%	110%	200%

- can be easily parallelized

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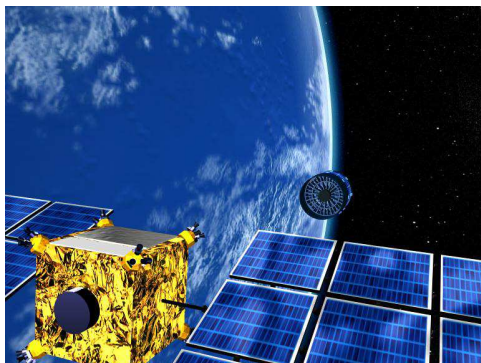
Mars Exploration Rover (MER) – ADCS Subsystem



- **A**ttitude **D**etermination and **C**ontrol **S**ystem (**ADCS**)
- 1-dimensional design problem, 30 choices
- complex uncertainty info, 34 dimensions



XEUS mission – permanent space-borne X-ray observatory



- complex design problem, 10 dimensions,
 $4 \times 14 \times 6 \times 8 \times 5 \times 20 \times 9 \times 44 \times 30 \geq 3 \cdot 10^9$
discrete choices, 1 continuous choice variable
- box uncertainty, 24 dimensions

Results

Number of simulation based worst-case estimations within close range (70% error) of linearization based worst-case searches:

	number of estimations	close results	percentage
MER	400	380	95.0%
XEUS	6204	5913	95.3%

Number of solutions close to the classical robust optimal solution with respect to different tolerances (sub := $\frac{\widehat{g}(\widehat{\theta}) - \widehat{g}(\widehat{\theta}_{\text{lin}})}{|\widehat{g}(\widehat{\theta}_{\text{lin}})|}$):

	# opt. runs	sub = 0	sub ≤ 5%	sub ≤ 10%	average sub
MER	10	1	1	8	7.6%
XEUS	10	0	10	10	2.4%

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Summary

- polyhedral clouds capture and process the uncertainty information available
- we embed clouds as the worst-case search in robust optimization
- high-dimensional worst-case search benefits from simulation based speed-up

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