Simulated Polyhedral Clouds in Robust Optimization

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Motivation

Challenges in real-life applications include:

- handling of uncertain parameters
  - → incomplete information
  - → high-dimensional uncertainties
- robust optimization
  - → bilevel formulation
  - → extra effort to account for robustness
  - → black box objective functions

1. Robust optimization
2. Potential clouds
3. Worst-case search
4. Applications
5. Summary

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Problem formulation

$$\min_{\theta \in \mathbf{T}} \max_{\varepsilon \in \mathcal{C}} g(\theta, \varepsilon) \tag{1}$$

- bilevel problem
- $\varepsilon \in \mathcal{C} \subseteq \mathbb{R}^n$ represents the uncertainties
- search space $\mathbf{T}$, typically a hyperinterval, possibly mixed integer variables
- black box objective function $g : \mathbf{T} \times \mathcal{C} \to \mathbb{R}$

Reformulation

Reformulate as two nested 1-level problems:

- outer level problem

$$\min_{\theta \in \mathbf{T}} \ \widehat{g}(\theta), \tag{2}$$

with

$$\widehat{g}(\theta) := g(\theta, \widehat{\varepsilon}), \tag{3}$$

and $\widehat{\varepsilon}$ the maximizer of the inner level problem

$$\max_{\varepsilon \in \mathcal{C}} \ g(\theta, \varepsilon), \tag{4}$$

for a given θ.

$\longrightarrow$ **worst-case search**

$\longrightarrow$ uncertainty modeling

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Potential clouds

- n-dimensional random vector ε
- potential function $V : \mathbb{R}^n \to \mathbb{R}$

Construct

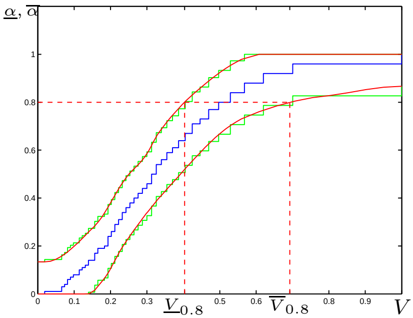
- lower α-cut $\underline{C}_\alpha := \{x \in \mathbb{R}^n \mid V(x) \leq \underline{V}_\alpha\}$
  contains at most a fraction of α of all possible scenarios
- upper α-cut $\overline{C}_\alpha := \{x \in \mathbb{R}^n \mid V(x) \leq \overline{V}_\alpha\}$
  contains at least a fraction of α of all possible scenarios

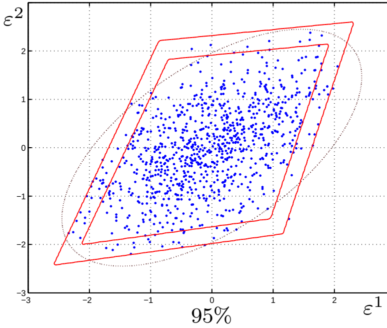
$\Rightarrow$ nested regions defining a **potential cloud**

How?

- regard $V(\varepsilon)$ as a 1-dimensional random variable
- find an enclosure of the CDF of $V(\varepsilon)$ (p-box)

Example





- level sets of V chosen polyhedral shaped
- α-cuts reasonably approximate the confidence regions linearly

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Challenges

The worst-case search is a 1-level problem for fixed $\theta \in \mathbf{T}$,

$$\max_{\varepsilon \in \mathcal{C}} \; g(\theta, \varepsilon). \tag{5}$$

- replaces nominal objective function of the non-robust formulation

$\longrightarrow$ produces overhead of treating uncertainties

- g can be computationally expensive, ε can be high-dimensional

$\longrightarrow$ worst-case search can be prohibitively expensive

$\Rightarrow$ speed-up required

Polyhedral uncertainty

- $f(\varepsilon) := g(\theta, \varepsilon)$ for fixed θ
- $\mathcal{C} := \{\varepsilon \mid A(\varepsilon - m) \leq b\}$, a polyhedral α-cut

$\Rightarrow$ worst-case search turns into

$$\max_{\varepsilon} \; f(\varepsilon) \quad \text{s.t.} \; A(\varepsilon - m) \leq b. \tag{6}$$

- classical approach: linearize f and solve LP
- simulation based approach for bound constraints $\varepsilon \in b_0$ and linear f: **Cauchy deviates method**

$$[\min_{\varepsilon} f(\varepsilon), \max_{\varepsilon} f(\varepsilon)] \quad \text{s.t.} \; \varepsilon \in b_0. \tag{7}$$

Modified Cauchy deviates method

1. evaluation at center
   - compute $f(m)$
2. sample $\mathcal{C}$ uniformly
   - rejection step
3. transformation to Cauchy distribution via inverse Cauchy CDF
   - sample point x_i possibly outside $\{\mathcal{C} - m\}$
4. normalization step
   - $K_i = \max_i \left(\frac{Ax_i}{b}\right) \Rightarrow \frac{x_i}{K_i} \in \{\mathcal{C} - m\}$
5. simulated deviation
   - $\delta_i = K_i(f_i - f(m))$, with $f_i := f\left(\frac{x_i}{K} + m\right)$
6. thus generate N sample points $\delta_1, \ldots, \delta_N$

Modified Cauchy deviates method ctd.

$\longrightarrow$ estimate the deviation Δ of f in $\mathcal{C}$ via max-likelihood from $\delta_1, \dots, \delta_N$

$\longrightarrow$ approximate solution

$$\max_{\varepsilon \in \mathcal{C}} f(\varepsilon) \approx f(m) + \Delta \tag{8}$$

- tractable estimation error even for very small N, useful if linearization cannot be afforded

N	800	200	100	50	20	10	5
error	10%	20%	30%	40%	70%	110%	200%

- can be easily parallelized

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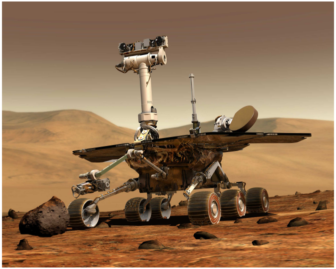
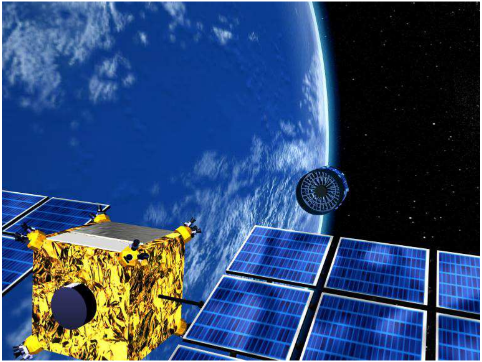
Mars Exploration Rover (MER) – ADCS Subsystem



- **A**ttitude **D**etermination and **C**ontrol **S**ystem (**ADCS**)
- 1-dimensional design problem, 30 choices
- complex uncertainty info, 34 dimensions

XEUS mission – permanent space-borne X-ray observatory



- complex design problem, 10 dimensions, $4 \times 14 \times 6 \times 8 \times 5 \times 20 \times 9 \times 44 \times 30 \geq 3 \cdot 10^9$ discrete choices, 1 continuous choice variable
- box uncertainty, 24 dimensions

Results

Number of simulation based worst-case estimations within close range (70% error) of linearization based worst-case searches:

	number of estimations	**close results**	**percentage**
MER	400	380	95.0%
XEUS	6204	5913	95.3%

Number of solutions close to the classical robust optimal solution with respect to different tolerances ($\mathrm{sub} := \frac{\widehat{g}(\widehat{\theta})-\widehat{g}(\widehat{\theta}_{\mathrm{lin}})}{|\widehat{g}(\widehat{\theta}_{\mathrm{lin}})|}$):

	# opt. runs	$\mathrm{sub} = 0$	$\mathrm{sub} \leq 5\%$	$\mathrm{sub} \leq 10\%$	**average** sub
MER	10	1	1	8	7.6%
XEUS	10	0	10	10	2.4%

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Summary

- polyhedral clouds capture and process the uncertainty information available
- we embed clouds as the worst-case search in robust optimization
- high-dimensional worst-case search benefits from simulation based speed-up

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