Robust optimization for aerospace applications

Martin Fuchs

CERFACS, Toulouse, France

September, 2011
Challenges in real-life applications include:

- handling of uncertain parameters
  - incomplete information
  - high-dimensional uncertainties

- robust optimization
  - bilevel formulation
  - extra effort to account for robustness
  - black box objective functions
1. Robust optimization

2. Potential clouds

3. Worst-case search

4. Aerospace applications

5. Summary
Robust optimization for aerospace applications

1. Robust optimization
2. Potential clouds
3. Worst-case search
4. Aerospace applications
5. Summary
Problem formulation

\[
\min_{\theta \in T} \ \max_{\varepsilon \in C} \ g(\theta, \varepsilon)
\]  

- bilevel problem
- \( \varepsilon \in C \subseteq \mathbb{R}^n \) represents the uncertainties
- search space \( T \), typically a hyperinterval, possibly mixed integer variables
- black box objective function \( g : T \times C \rightarrow \mathbb{R} \)
Reformulation

Reformulate as two nested 1-level problems:

- **outer level problem**
  \[
  \min_{\theta \in T} \hat{g}(\theta), \tag{2}
  \]
  
with

\[
\hat{g}(\theta) := g(\theta, \hat{\varepsilon}), \tag{3}
\]

and \(\hat{\varepsilon}\) the maximizer of the inner level problem

\[
\max_{\varepsilon \in \mathcal{C}} g(\theta, \varepsilon), \tag{4}
\]

for a given \(\theta\).

→ **worst-case search**

→ uncertainty modeling
Robust optimization for aerospace applications
Potential clouds

- $n$-dimensional random vector $\varepsilon$
- potential function $V : \mathbb{R}^n \rightarrow \mathbb{R}$

Construct

- lower $\alpha$-cut $C_\alpha := \{ x \in \mathbb{R}^n \mid V(x) \leq V_\alpha \}$
  contains at most a fraction of $\alpha$ of all possible scenarios
- upper $\alpha$-cut $\overline{C}_\alpha := \{ x \in \mathbb{R}^n \mid V(x) \leq \overline{V}_\alpha \}$
  contains at least a fraction of $\alpha$ of all possible scenarios

$\Rightarrow$ nested regions defining a **potential cloud**

How?

- regard $V(\varepsilon)$ as a 1-dimensional random variable
- find an enclosure of the CDF of $V(\varepsilon)$ (p-box)
**Example**

- Level sets of $V$ chosen polyhedral shaped
- $\alpha$-cuts reasonably approximate the confidence regions linearly
<table>
<thead>
<tr>
<th></th>
<th>Robust optimization</th>
<th>Potential clouds</th>
<th>Worst-case search</th>
<th>Aerospace applications</th>
<th>Summary</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Robust optimization</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Potential clouds</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Worst-case search</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Aerospace applications</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Summary</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Challenges

The worst-case search is a 1-level problem for fixed $\theta \in T$,

$$\max_{\varepsilon \in C} g(\theta, \varepsilon).$$  \hspace{1cm} (5)

- replaces nominal objective function of the non-robust formulation

$\rightarrow$ produces overhead of treating uncertainties

- $g$ can be computationally expensive,
  $\varepsilon$ can be high-dimensional

$\rightarrow$ worst-case search can be prohibitively expensive

$\Rightarrow$ speed-up required
Models for $g$

Possible models for $g$ mainly depend on the number $N$ of evaluations available, e.g.,

- $N = \frac{(n+2)(n+1)}{2} \ldots$ quadratic model

- $n + 1 < N < \frac{(n+2)(n+1)}{2}$
  $\ldots$ Minimum Frobenius norm model

- $N = n + 1 \ldots$ linear model

- $N < n + 1 \ldots$ ?
Polyhedral uncertainty

- \( f(\varepsilon) := g(\theta, \varepsilon) \) for fixed \( \theta \)
- \( \mathcal{C} := \{ \varepsilon | A(\varepsilon - m) \leq b \} \), a polyhedral \( \alpha \)-cut

\( \Rightarrow \) worst-case search turns into

\[
\max_{\varepsilon} f(\varepsilon) \\
\text{s.t. } A(\varepsilon - m) \leq b.
\] (6)

- former approach: linearize \( f \) and solve LP
- simulation based approach for bound constraints \( \varepsilon \in b_0 \) and linear \( f \): \textbf{Cauchy deviates method}

\[
[\min_{\varepsilon} f(\varepsilon), \max_{\varepsilon} f(\varepsilon)]
\] (7)

\text{s.t. } \varepsilon \in b_0.
Modified Cauchy deviates method

1. evaluation at center
   - compute $f(m)$

2. sample $C$ uniformly
   - rejection step

3. transformation to Cauchy distribution via inverse Cauchy CDF
   - sample point $x_i$ possibly outside $\{C - m\}$

4. normalization step
   - $K_i = \max_i \left( \frac{Ax_i}{b} \right) \Rightarrow \frac{x_i}{K_i} \in \{C - m\}$

5. simulated deviation
   - $\delta_i = K_i(f_i - f(m))$, with $f_i := f\left(\frac{x_i}{K} + m\right)$

6. thus generate $N$ sample points $\delta_1, \ldots, \delta_N$
Modified Cauchy deviates method ctd.

\[ \rightarrow \text{estimate the deviation } \Delta \text{ of } f \text{ in } C \text{ via max-likelihood from } \delta_1, \ldots, \delta_N \]

\[ \rightarrow \text{approximate solution} \]

\[
\max_{\varepsilon \in C} f(\varepsilon) \approx f(m) + \Delta \tag{8}
\]

- tractable estimation error even for very small \( N \), useful if linearization cannot be afforded

<table>
<thead>
<tr>
<th>( N )</th>
<th>800</th>
<th>200</th>
<th>100</th>
<th>50</th>
<th>20</th>
<th>10</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>error</td>
<td>10%</td>
<td>20%</td>
<td>30%</td>
<td>40%</td>
<td>70%</td>
<td>110%</td>
<td>200%</td>
</tr>
</tbody>
</table>

- can be easily parallelized
1. Robust optimization

2. Potential clouds

3. Worst-case search

4. Aerospace applications

5. Summary

Robust optimization for aerospace applications
I. Mars Exploration Rover (MER) – ADCS Subsystem

- **Attitude Determination and Control System (ADCS)**
- 1-dimensional design problem, 30 choices
- complex uncertainty info, 34 dimensions
II. XEUS mission – permanent space-borne X-ray observatory

- complex design problem, 10 dimensions,
  \[4 \times 14 \times 6 \times 8 \times 5 \times 20 \times 9 \times 44 \times 30 \geq 3 \times 10^9\]
- discrete choices, 1 continuous choice variable
- box uncertainty, 24 dimensions
Results

Number of simulation based worst-case estimations within close range (70\% error) of linearization based worst-case searches:

<table>
<thead>
<tr>
<th>number of estimations</th>
<th>close results</th>
<th>percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>MER</td>
<td>400</td>
<td>380</td>
</tr>
<tr>
<td>XEUS</td>
<td>6204</td>
<td>5913</td>
</tr>
</tbody>
</table>

Number of solutions close to the former robust optimal solution with respect to different tolerances (\(\text{sub} := \frac{\hat{g}(\hat{\theta}) - \hat{g}(\hat{\theta}_{\text{lin}})}{|\hat{g}(\hat{\theta}_{\text{lin}})|}\)):

<table>
<thead>
<tr>
<th># opt. runs</th>
<th>sub = 0</th>
<th>sub \leq 5%</th>
<th>sub \leq 10%</th>
<th>average sub</th>
</tr>
</thead>
<tbody>
<tr>
<td>MER</td>
<td>10</td>
<td>1</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>XEUS</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>
III. Aircraft wing shape optimization

- design problem in 12 dimensions (wing shape is represented by 6 bumps on a 2D profile)
- box uncertainty in 20 dimensions (e.g., flight conditions)
Particularities

- the black box objective is to minimize the drag of the wing and contains a penalty term if the lift is too small

- one black box call takes up to 40 min. of computation time → we chose $N = 10$, and 40 iterations in the outer level for one short run of 9 days

- the black box is called remotely → need a framework with interfaces between, e.g., Matlab solvers and the black box calls

- no structure information about the black box available (e.g., separability)
Result visualizations

- Deformation field of an optimized wing (scaled).
Result visualizations ctd.

- Two Mach-flows for an optimized wing: Nominal case (left) and worst case (right).
1. Robust optimization
2. Potential clouds
3. Worst-case search
4. Aerospace applications
5. Summary
Summary

- polyhedral clouds capture and process the uncertainty information available
- we embed clouds as the worst-case search in robust optimization
- high-dimensional worst-case search benefits from simulation based speed-up

Visit my website: http://www.martin-fuchs.net