Higher dimensional uncertainty modeling with polyhedral clouds

Martin Fuchs

CERFACS, Toulouse, France

April 12, 2010

Goals

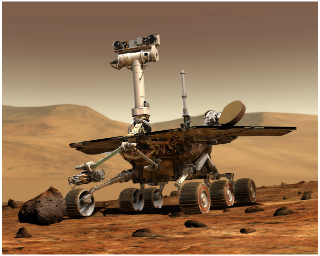
Cope with

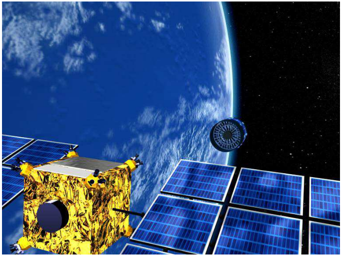
→ real-life applications

- Curse of dimensionality
  - severe computational effort
- Incomplete information
  - scarce data, conflicting, or unformalized information
  - typically available:
    intervals, marginal CDFs, information updates
  - typically **not** available:
    correlation information, sufficient amount of data
- Unjustified assumptions

Overview

- Uncertainty modeling with clouds
- Robust design optimization
- Applications: Spacecraft design





- Summary

Uncertainty model – basic idea

- gather available uncertainty information
- choose an admissible failure probability p_f
- determine bounds on confidence regions
  - contain at least $1 - p_f$ of all possible scenarios $\varepsilon \in \mathbb{M} \subseteq \mathbb{R}^n$
  - could be computed from bounds on the CDF of ε, **but**: curse of dimensionality & lack of information
  - powerful tool in 1D: Kolmogorov-Smirnov (KS) statistics for empirical data

$\Rightarrow$ simulate data if no real data available

$\Rightarrow$ reduce the dimension to 1 by means of a potential function $V : \mathbb{R}^n \to \mathbb{R}$

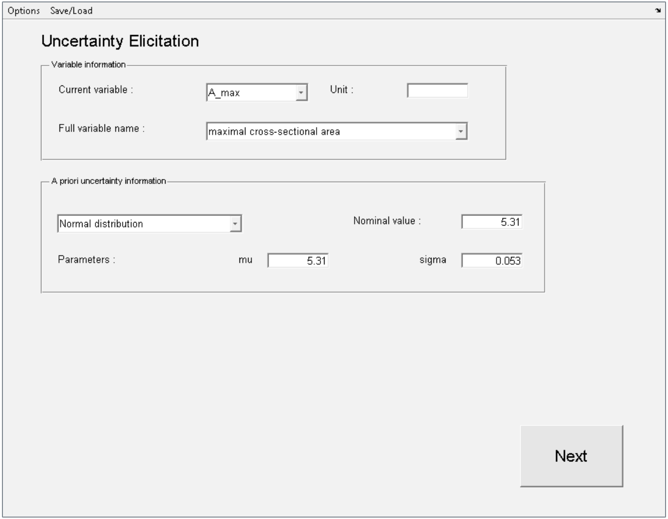
$\Rightarrow$ apply KS as in 1D case to get bounds on the CDF of $V(\varepsilon)$

$\Rightarrow$ lower and upper confidence regions for $V(\varepsilon)$

$\Rightarrow$ lower and upper confidence regions for ε as level sets of V

$\Rightarrow$ **potential cloud**

A priori information



Sample generation

1. Simulate sample with respect to the a priori uncertainty information

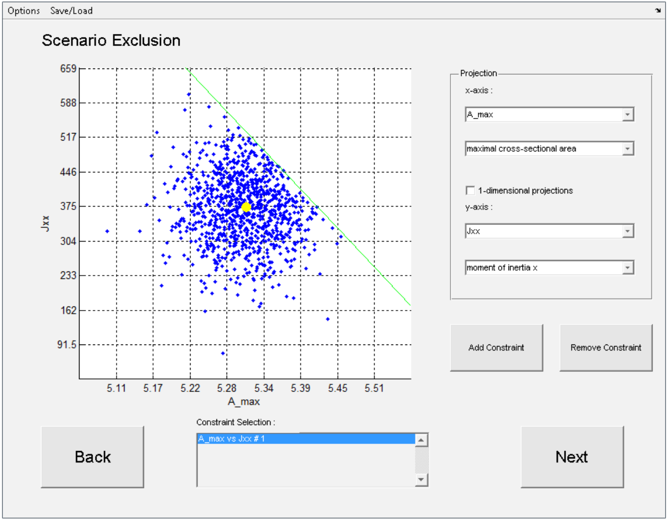
   → Latin hypercube sampling

2. Modify the simulated sample

   - add subjective, unformalized knowledge
   - linear scenario exclusion in 1D or 2D projections

     → polyhedral constraints

Linear scenario exclusion



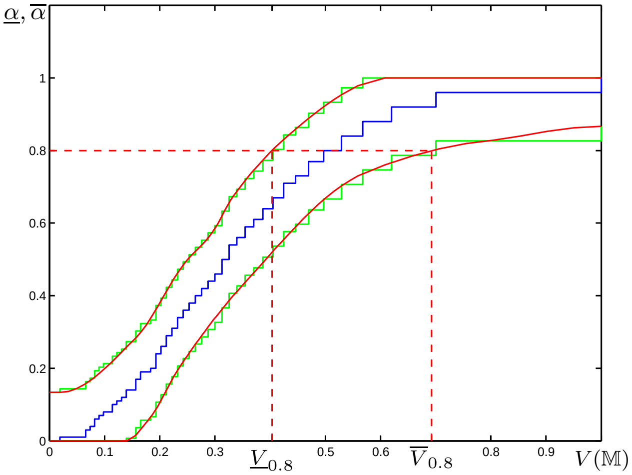
Probability bounds

- process towards probability bounds
- do **not** bound the CDF of $\varepsilon \in \mathbb{R}^n$
- **KS bounds** of the CDF of the potential $V(\varepsilon)$, i.e., $\widetilde{F} \pm d_{\mathrm{KS}}$ with

$$d_{\mathrm{KS}} = \frac{\phi^{-1}(\alpha_{\mathrm{KS}})}{\sqrt{N} + 0.12 + \frac{0.11}{\sqrt{N}}}$$

  - $\widetilde{F}$ the empirical CDF of $V(\varepsilon)$
  - ϕ the Kolmogorov function
  - α_{KS} the confidence in the KS theorem

Probability bounds ctd.



Polyhedral clouds summary

- n-dimensional random vector ε, potential $V : \mathbb{R}^n \to \mathbb{R}$
  - our choice of V: box shaped from the a priori information
  - add the subjective cutoffs

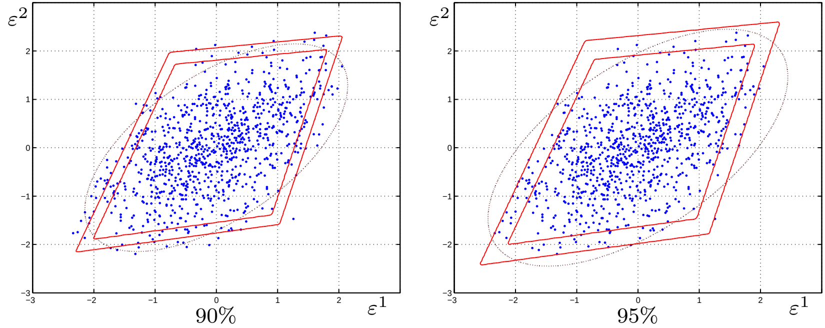
  $\Rightarrow$ **polyhedron shape**

- lower α-cut $\underline{C}_\alpha := \{x \in \mathbb{R}^n \mid V(x) \leq \underline{V}_\alpha\}$
  contains at most a fraction of α of all possible scenarios

- upper α-cut $\overline{C}_\alpha := \{x \in \mathbb{R}^n \mid V(x) \leq \overline{V}_\alpha\}$
  contains at least a fraction of α of all possible scenarios

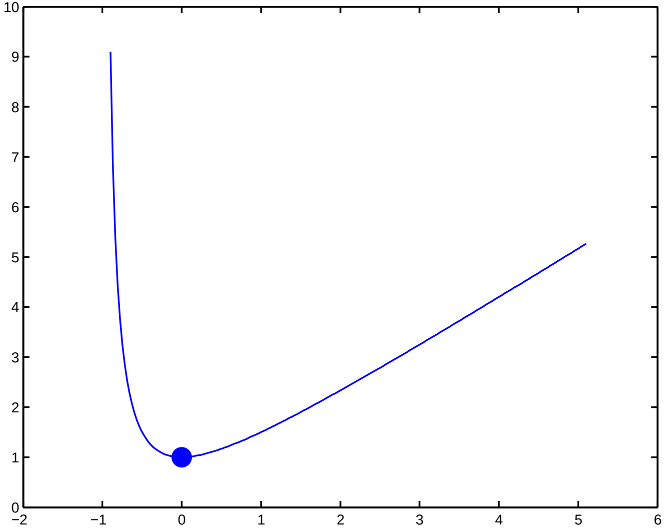
$\Rightarrow$ nested regions defining a **polyhedral cloud**

Example

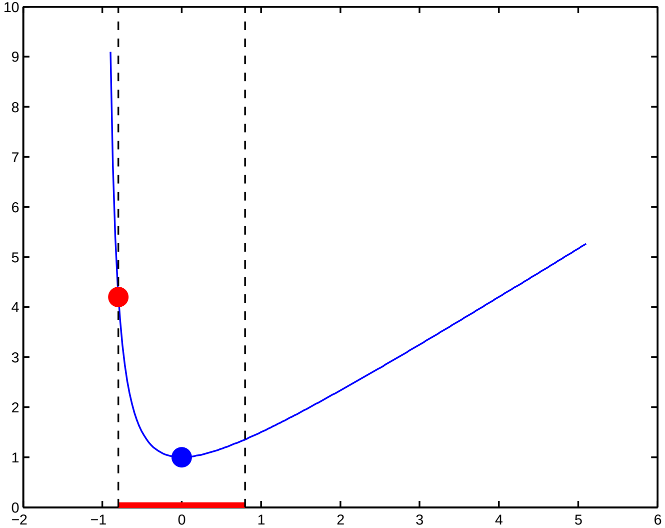


- assume that the probability distribution is hidden, an expert only knows the data
- the expert may have knowledge about the physical dependence of the variables
- polyhedral constraints model this knowledge, i.e., α-cuts reasonably approximate the confidence regions linearly

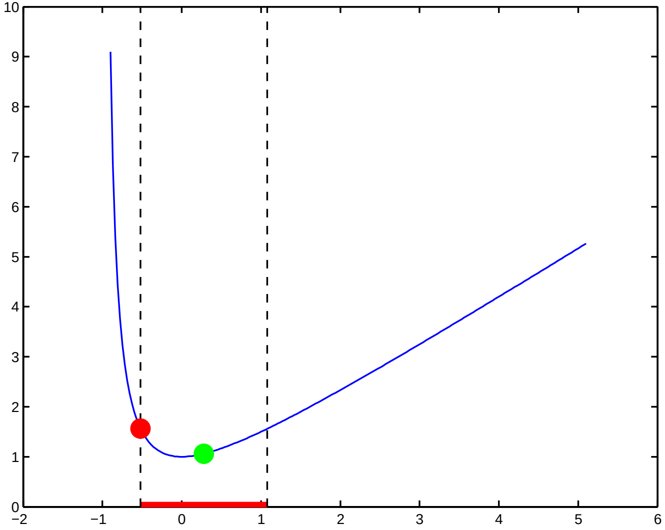
Non-robust optimization



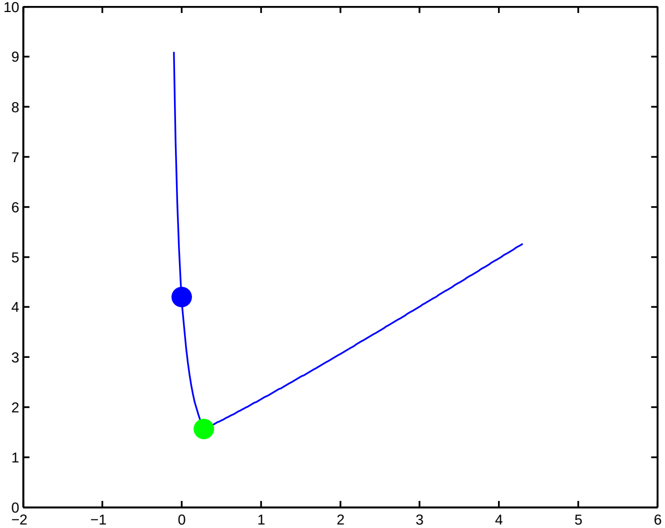
Critical worst-case



Robust optimization



Worst-case function



Worst-case search with clouds

- choose confidence level α
- cloud given by the polyhedra $\underline{C}_\alpha$, $\overline{C}_\alpha$
- search for the worst-case scenario in $\underline{C}_\alpha$ or $\overline{C}_\alpha$:

$$\max_{\varepsilon \in \underline{C}_\alpha} g(\theta, \varepsilon), \tag{1}$$

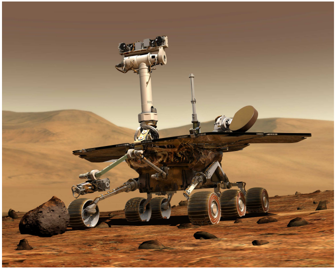
  - θ, n_0-dimensional design point (fixed)
  - $g : \mathbb{R}^m \to \mathbb{R}$, design objective (sought to be minimal)

Robust design optimization

$$\min_{\theta \in \mathbf{T}} \max_{\varepsilon \in \underline{C}_\alpha} g(\theta, \varepsilon) \tag{2}$$

- bilevel problem
- nonlinear or black box objective function $g : \mathbf{T} \times \underline{C}_\alpha \to \mathbb{R}$
- mixed integer programming (depending on the search space $\mathbf{T}$)

Mars Exploration Rover Subsystem



- **A**ttitude **D**etermination and **C**ontrol **S**ystem (**ADCS**)
- 1-dimensional design problem, only 30 choices
- complex uncertainty info, 34 dimensions

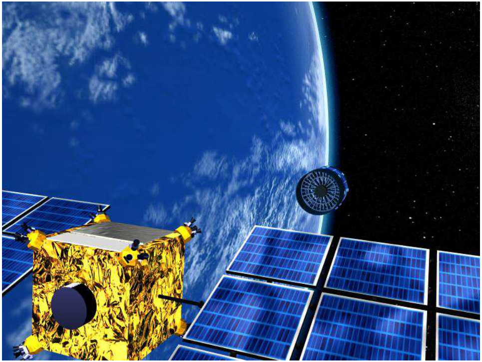
Optimization results

- Minimize the objective function: total mass m_{tot}
- Optimization on the nominal values:
  Optimal design choice: $\theta = 3$
- Optimization of the worst case:
  Optimal design choice: $\theta = 9$

θ	**nominal case** m_{tot}	**worst case** m_{tot}
3	2.68	8.75
9	3.24	8.08

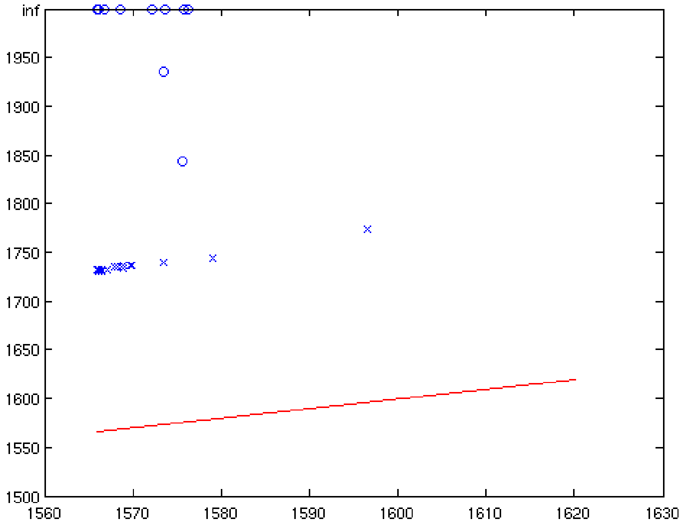
$\Rightarrow$ design point sensitive to accounting for uncertainty

XEUS mission – permanent space-borne X-ray observatory



- complex design problem, 10 dimensions, $4 \times 14 \times 6 \times 8 \times 5 \times 20 \times 9 \times 44 \times 30 \geq 3 \cdot 10^9$ discrete choices, 1 continuous choice variable
- box uncertainty, 24 dimensions

Optimization results



Summary

Uncertainty modeling with polyhedral clouds enables to

- capture and process incomplete and unformalized information
- allow for an intuitive uncertainty elicitation and information updating
- solve successfully higher dimensional real-life problems of robust design optimization

Visit my website: `http://www.martin-fuchs.net`